The structure of Witt-Burnside rings

Witt vectors, arithmetic, geometry, and topology

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We’ve already seen and will many great talks on how ubiquitous and important Witt vectors are. This talk will focus on the algebraic properties of their generalizations.

- Review the Dress and Siebeneicher’s generalization involving profinite groups.
- Topology and ideals
- Prime ideals
Introduction  Generalized Witt Vectors  (Non)-Finite generation and topology  Prime Ideals

Review of $p$-typical Witt Vectors

In $\mathbf{W}(A) = \{(a_0, a_1, \ldots) : a_i \in A\}$, the ring operations are

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (s_0(a_0; b_0), s_1(a_0, a_1; b_0, b_1), \ldots),$$

$$(a_0, a_1, \ldots) \cdot (b_0, b_1, \ldots) = (m_0(a_0; b_0), m_1(a_0, a_1; b_0, b_1), \ldots),$$

where the polynomials $s_i$ and $m_i$ are defined via Witt polynomials

$$W_n(x_0, \ldots, x_n) = \sum_{i=0}^{n} p^i x_i^{p^{n-i}} = x_0^{p^n} + px_1^{p^{n-1}} + \cdots + p^n x_n$$
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by solving for $s = (s_0, s_1, \ldots)$ and $m = (m_0, m_1, \ldots)$ in equations

$$W_n(x) + W_n(y) = W_n(s), \quad W_n(x) W_n(y) = W_n(m)$$

for all $n \geq 0$. The $s_i$'s and $m_i$'s easily have $\mathbb{Z}[1/p]$-coefficients, but in fact have $\mathbb{Z}$-coefficients (Witt).
Generalized Witt Vectors using Profinite Groups

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The natural **indices** of the variables for these polynomials are the **discrete transitive** $G$-sets up to $G$-set isomorphism. Concretely these are coset spaces $G/H$ with **open** $H$ (up to conjugation) and are called the **frame of** $G$. 
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**Example.** $G = \mathbb{Z}_p$. The discrete transitive $G$-sets are $\mathbb{Z}_p/p^n\mathbb{Z}_p$ for $n \geq 0$. The frame of $\mathbb{Z}_p$ is like $\mathbb{N} = \{0, 1, 2, \ldots \}$ or $\{1, p, p^2, \ldots \}$. The variables $x_0, x_1, \ldots$ in the classical Witt polynomials are indexed by $\mathbb{N}$. 
There is a natural **partial ordering** on the frame of $G$: say $U \leq T$ when there is a $G$-map $T \to U$ (concretely, $G/H \leq G/K$ when $K \subset gHg^{-1}$ for some $g \in G$). The minimal point is $0 = G/G$. 

**Example.** $G = \mathbb{Z}/p$. We have $\mathbb{Z}/p/\mathbb{Z}/p^n \leq \mathbb{Z}/p/\mathbb{Z}/p^m$ when $n \leq m$, so even as a partially ordered set the frame of $\mathbb{Z}/p$ corresponds to $\mathbb{N} = \{0, 1, 2, \ldots\}$ or to $\{1, p, p^2, \ldots\}$ (with the latter ordered by divisibility).
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It will be helpful to have a picture of the frame of $G$ as a graph using the partial ordering. Here is the frame of $G = \mathbb{Z}_2$.

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More generally, the frames of $\mathbb{Z}_p$ or $\mathbb{Z}/p^n\mathbb{Z}$ are totally ordered, but the frame of $\mathbb{Z}_p^2$ is not totally ordered...
The frame of $G = \mathbb{Z}_2^2$ is much more intricate than for $\mathbb{Z}_2$!

Here $H = \mathbb{Z}_2^2(1_0) + \mathbb{Z}_2^2(0_2)$ and $K = \mathbb{Z}_2^2(1_0) + \mathbb{Z}_2^2(0_4)$. Since $K \subset H$, we have $G/H \leq G/K$. 
Visualizing the Frame of $\mathbb{Z}_2^2$

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Note: For pro-$p$ $G$ besides $\mathbb{Z}_p$ and $\mathbb{Z}/p^n\mathbb{Z}$, there is more than one (open) subgroup of index $p$, so multiple vertices connect to the trivial $G$-set $G/G$. This is a crucial aspect which accounts for the change of structure from $d = 1$ to $d > 1$. 
Defining Generalized Witt Polynomials

For any discrete transitive $G$-set $T$, define the $T$-th Witt polynomial in $\mathbb{Q}[\ldots, x_T, \ldots] = \mathbb{Q}[x]$ to be

$$W_T(x) = \sum_{U \leq T} \#\text{Map}_G(T, U)x_U^{#T/#U} = x_0^{#T} + \cdots + #\text{Aut}_G(T)x_T.$$
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Therefore given independent variables $x = (x_T)$ and $y = (y_T)$, in $\mathbb{Q}[x, y]$ there are unique polynomial sequences $s = (s_T(x, y))$ and $m = (m_T(x, y))$ satisfying

$$W_T(x) + W_T(y) = W_T(s), \quad W_T(x)W_T(y) = W_T(m)$$

for all discrete transitive $G$-sets $T$. 

These polynomials $s_T$ and $m_T$ were defined by Dress and Siebeneicher (1988), who showed they have $\mathbb{Z}$-coefficients.
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Dress and Siebeneicher defined the ring of generalized Witt vectors \( W_G(A) \) for any commutative ring \( A \) as the \( A \)-valued sequences \( a = (a_T) \) indexed by the frame of \( G \), with operations

\[ a + b = (s_T(a, b)), \quad ab = (m_T(a, b)). \]

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The Problem with Witt Vectors

Witt vector ring operations are a **nightmare** to work with explicitly.
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The formulas do not fit in the head of a civilized mathematician of the 21st century. Hendrik Lenstra
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In contrast, for $G \not\cong Z_p$, there can be many topologies for $\mathbf{W}_G(k)$. 
Structures and topologies

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In contrast, for $G \ncong \mathbb{Z}_p$, there can be many topologies for $\mathbf{W}_G(k)$.

For open normal subgroups $N \subset N' \subset G$, there are natural projection maps $\mathbf{W}_{G/N}(k) \to \mathbf{W}_{G/N'}(k)$ forming a projective system. Giving each factor $\mathbf{W}_{G/N}(k)$ the discrete topology induces a topology on $\mathbf{W}_G(k) \cong \varprojlim_N \mathbf{W}_{G/N}(k)$ where $N$ ranges over the open normal subgroups of $G$. We refer to this as the profinite topology and $\mathbf{W}_G(k)$ is complete in it.
A new topology?

For classic Witt vectors, the profinite topology agrees with the topology induced by the maximal ideal \( m \). For perfect \( k \), one has

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\mathbf{a} \in m^n = (p^n) \text{ if and only if } a_i = 0 \text{ for } i < n.
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For any profinite group \( G \), one is inclined to consider for any profinite \( G \) the ideals

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I_n(G, k) = \{ \mathbf{a} \in \mathbf{W}_G(k): a_T = 0 \text{ for } \# T < n \}.
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Obviously

$$\mathcal{W}_G(k) = I_1(G, k) \supset I_2(G, k) \supset \cdots \supset I_n(G, k) \supset I_{n+1}(G, k) \supset \cdots,$$

and $\bigcap I_n(G, k) = \{0\}$. 

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$$W_G(k) = I_1(G, k) \supset I_2(G, k) \supset \cdots \supset I_n(G, k) \supset I_{n+1}(G, k) \supset \cdots,$$

and $\bigcap I_n(G, k) = \{0\}$. So these ideal define a fundamental system of neighborhoods of $0$, giving a Hausdorff topology on $W_G(k)$ called the *initial vanishing topology*. It is easy to see that $W_G(k)$ is complete with respect to this topology.
Example of $I_n(G, k)$ for $G = \mathbb{Z}_2^2$

One can easily visualize the ideals $I_n(G, k)$. For example when $G = \mathbb{Z}_2^2$:  

![Diagram of $I_2(G, k)$]
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$$
\begin{array}{c|cccc}
\text{Size} & 1 & 2 & 4 & 8 \\
I_8(G, k) & & & & \\
\end{array}
$$
Comparing topologies

How do these topologies relate?

Theorem
Let $G$ be a topologically finitely generated profinite group. The profinite and initial vanishing topologies agree.

The proof compares the families of ideals $\{I_n\}_{n \geq 1}$ and $\{K_N\}_{N \text{ open}}$ where $K_N$ is the kernel of the natural projection map $\text{Proj}_G \ G / N : \mathbb{W}_G(k) \to \mathbb{W}_G / N(k)$ defined by $a \mapsto (a^T) T \in F(G / N)$.

The key fact is that for topologically finitely generated profinite groups, there are finitely many open subgroups of each index. The converse also holds when $G$ is pro-$p$. 
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The converse also holds when \( G \) is pro-\( p \).
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When \( G \) is (not necessarily) abelian, and pro-\( p \) a similar condition that can be shown to always hold. In particular for \( G \) pro-\( p \):

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I_p(G, k)I_{p^n}(G, k) \subseteq I_{p^{n+1}}(G, k).
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It is natural to ask if \( l_m(G, k)l_n(G, k) \subset l_{m+n}(G, k) \)? This is easy to show if \( G \) is abelian.

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So, \( l_p(G, k)^m \subset l_{p^m}(G, k) \) for any \( m \geq 1 \). This gives special importance to \( m := l_p(G, k) \).
The Maximal Ideal in $\mathbf{W}_G(k)$

Like the classical case, we have

**Theorem (M)**

*When $G$ an infinite pro-$p$ group and $k$ is a field of characteristic $p$, $\mathbf{W}_G(k)$ is a local ring with maximal ideal $m = I_p(G, k) = \{(0, *, *, \ldots)\}$.*
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Indeed, given a Witt vector $a = (a_0, *, *, \ldots)$ with $a_0 \neq 0$, we can multiply it by the unit $(a_0^{-1}, 0, 0, \ldots)$ to assume $a_0 = 1$. Then our Witt vector is

$$(1, *, *, \ldots) = (1, 0, 0, \ldots) + (0, *, *, \ldots).$$
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Powers of the second term tend to 0 in the initial vanishing topology on $\mathbf{W}_G(k)$, so we can form an inverse using geometric series.
Strutures and topologies

So when $G$ is pro-$p$, there are four topologies:

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2. the $m$-adic topology,
3. the initial vanishing topology, and
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We have answered how (3) and (4) relate. Furthermore, the inclusion of ideals $(p)^n \subset m^n \subset I_{p^n}(G, k)$ induce comparisons between these topologies. We will see that they will differ in general because these contains cannot be reversed.
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To see this we examine properties of elements of $(p)$ and $\mathfrak{m}$ in $W_G(k)$ for $G = \mathbb{Z}_p^d$ for $d > 1$. 
An element of $\mathcal{W}_{\mathbb{Z}_2} (\mathbb{F}_2) = \mathbb{Z}_2$ is a 0 or 1 at each vertex. Here is what 1 and 2 look like in this ring.

```
1  0  0  0  0  0  0  0  0
  ●●●●●●●●●●●●

0  1  0  0  0  0  0  0  0
  ●●●●●●●●●●●●
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Visualizing Elements in $W_{\mathbb{Z}^2}(\mathbb{F}_2)$

An element of $W_{\mathbb{Z}^2}(\mathbb{F}_2)$ is a 0 or 1 at each vertex. Here is the representation of 1 in this ring.
Visualizing Elements in $\mathbf{W}_{\mathbb{Z}_2^2}(\mathbb{F}_2)$

Here is 2 in $\mathbf{W}_{\mathbb{Z}_2^2}(\mathbb{F}_2)$. It is quite different from the classical case ($G = \mathbb{Z}_2$): infinitely many nonzero coordinates.
Visualizing Elements in $\mathbf{W}_{Z^2_p}(\mathbf{F}_p)$

However for odd $p$, the element $p$ in $\mathbf{W}_{Z^2_p}(\mathbf{F}_p)$ has finite support. For example, when $p = 3$: 
Multiplication by $p$

If $G$ is infinite pro-$p$ and not $\mathbb{Z}_p$, the multiple discrete transitive $G$-sets of size $p$ – the vertices linked to $G/G$ in the frame of $G$ – lead to zero divisors in $W_G(k)$: it is not a domain. (This construction is essentially in the work of Dress and Siebeneicher.)
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In $\mathbf{W}(k)$, $pa = (0, a_0^p, *, *, \ldots)$. An important effect of multiple $G$-sets of size $p$ is that $pa = (0, a_0^p, a_0^p, \ldots, a_0^p, *, *, \ldots)$, where $a_0^p$ occurs in all coordinates indexed by $T$ with size $p$: we have repeated coordinates.
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When $k$ is perfect of characteristic $p$, $\mathbf{W}(k) = \mathbf{W}_{\mathbb{Z}_p}(k)$ is a DVR with maximal ideal $(p)$. In $\mathbf{W}_G(k)$ with $G \not\cong \mathbb{Z}_p$, $(p) \subsetneq m$ from the repetitions.
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When $k$ is perfect of characteristic $p$, $\mathbf{W}(k) = \mathbf{W}_{\mathbb{Z}_p}(k)$ is a DVR with maximal ideal $(p)$. In $\mathbf{W}_G(k)$ with $G \not\cong \mathbb{Z}_p$, $(p) \subsetneq \mathfrak{m}$ from the repetitions. It wouldn’t be a surprise that $\mathfrak{m}$ is nonprincipal: $k[[x_1]]$ is a PID but $k[[x_1, x_2, \ldots, x_n]]$ is not if $n > 1$. 
If $G$ is a topologically finitely generated pro-$p$ group, surely $W_G(k)$ is Noetherian with maximal ideal $m$ generated by Witt vectors with a 1 in a coordinate indexed by $T$'s with size $p$. 
The Maximal Ideal in $\mathcal{W}_{Z_p^d}(k)$

If $G$ is a topologically finitely generated pro-$p$ group, surely $\mathcal{W}_G(k)$ is Noetherian with maximal ideal $m$ generated by Witt vectors with a 1 in a coordinate indexed by $T$'s with size $p$. This is wrong!
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**Theorem (M)**

If $G = \mathbb{Z}^d_p$ for $d \geq 2$ and $k$ is any field of characteristic $p$, the maximal ideal $m$ of $W_G(k)$ is not finitely generated. In fact, $\dim_k(m/m^2) = \infty$. 
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To prove this one notes all elements in $m^2$ have infinitely many repetitions in their coordinates. For each $n \geq 1$, there are $T_n$ and $T'_n$ in the frame of $\mathbb{Z}_p^d$ with size $p^n$ such that any element of $m^2$ has the same $T_n$ and $T'_n$ coordinates (need $d \geq 2$).
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Set $a_n = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ for $n \geq 1$, where the single 1 is in the $T_n$-slot. Then $a_n \in m$ and $a_n \not\in m^2$ by looking at the $T_n$ and $T'_n$ coordinates (this needs $d \geq 2$; $a_n \in m^2$ if $d = 1$ and $n \geq 2$).
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One can show that $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $k$. Since $n$ is arbitrary, this shows that $\mathfrak{m}$ is not finitely generated.
The Maximal Ideal in $\mathcal{W}_{d}(k)$

**Theorem (M)**

If $G = \mathbb{Z}_{p}^{d}$ for $d \geq 2$ and $k$ is any field of characteristic $p$, the maximal ideal $m$ of $\mathcal{W}_{G}(k)$ is not finitely generated. In fact, $\dim_{k}(m/m^{2}) = \infty$.

Another consequence of the repetitions in $m^{2}$ is that $m^{2}$ cannot contain $I_{pr}(\mathbb{Z}_{p}^{d}, k)$. Hence the $m$-adic topology and the initial vanishing topology differs.
Beyond Being non-Noetherian

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**Theorem (M)**

\[ \text{When } p \neq 2, \mathbf{W}_{\mathbb{Z}_p^2}(k) \text{ is not coherent when } k \text{ is a field of char. } p. \]
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When \( p \neq 2 \), \( W_{Z_p}^2(k) \) is not coherent when \( k \) is a field of char. \( p \).

Sketch of proof: There is a specific Witt vector whose annihilator ideal is studied.

If this ideal has \( r \) generators, for any \( n \geq 1 \) a computation in the ideal using Witt vector coordinates indexed by the special \( G \)-sets \( T_1, T'_1, \ldots, T_n, T'_n \) mentioned before leads to a surjective polynomial map \( k^r \rightarrow k^n \).
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*When* $p \neq 2$, $W_{Z_p^2}(k)$ *is not coherent when* $k$ *is a field of char. $p$.*

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Take $n > r$ to have a contradiction!
Prime Ideals in $W_{\mathbb{Z}_p^2}(k)$

The ring $W_G(k)$ can be non-Noetherian, not a domain, and not coherent. Does it have any basic positive properties?
Prime Ideals in $\mathbf{W}_{\mathbb{Z}_p^2}(k)$

The ring $\mathbf{W}_{\mathbb{Z}_p^2}(k)$ can be non-Noetherian, not a domain, and not coherent. Does it have any basic positive properties? Another “$p$-adic” ring which is non-Noetherian and not a domain is the continuous functions $\mathbb{Z}_p \rightarrow \mathbb{Q}_p$, which is a reduced ring.
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Given nonzero $\mathbf{v}$ in $\mathbf{W}_{\mathbb{Z}_p^2}(k)$ we pick its “smallest” nonzero coordinate and find a predictable nonzero coordinate for powers of $\mathbf{v}$. This uses the combinatorial structure of the frame of $\mathbb{Z}_p^2$: the lowest part of it *forms a tree*. It is *not a tree* for $\mathbb{Z}_p^d$ when $d > 2$. 
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Perhaps $W_{\mathbb{Z}_p^d}(k)$ is reduced for $d \geq 3$, but a new proof is needed.
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Since $W_{Z_p^2}(k)$ is reduced, the intersection of its prime ideals is $\{0\}$. Thus $W_{Z_p^2}(k)$ embeds into $\prod_P W_{Z_p^2}(k)/P$, a product of domains, via reduction mod $P$ for all $P$:

$$W_{Z_p^2}(k) \to \prod_P W_{Z_p^2}(k)/P$$

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It would be nice if we could determine all the domains $\mathbb{W}_{\mathbb{Z}_p^2}(k)/P$ concretely and then determine how $\mathbb{W}_{\mathbb{Z}_p^2}(k)$ sits inside this product ring to get a more concrete idea of what $\mathbb{W}_{\mathbb{Z}_p^2}(k)$ “is”.
Prime Ideals in $\mathbf{W}_{\mathbb{Z}_p}(k)$

There is a natural family of prime ideals in $\mathbf{W}_{\mathbb{Z}_p}(k)$. In the diagram, we illustrate with $p = 2$. 

![Diagram showing levels and prime ideals in $\mathbf{W}_{\mathbb{Z}_p}(k)$]

The image is a domain, so the kernel is a prime ideal. Each path along the bottom gives a different prime ideal in $\mathbf{W}_{\mathbb{Z}_p}(k)$. Are these all the prime ideals (besides the maximal ideal)?
Prime Ideals in $\mathbf{W}_{\mathbb{Z}_p^2}(k)$

There is a natural family of prime ideals in $\mathbf{W}_{\mathbb{Z}_p^2}(k)$. In the diagram, we illustrate with $p = 2$. Choose a level 0 path $C$. Projecting onto these coordinates is a surjective homomorphism $\mathbf{W}_{\mathbb{Z}_p^2}(k) \to \mathbf{W}(k)$. 
There is a natural family of prime ideals in $\mathbf{W}_{\mathbb{Z}_p}(k)$. In the diagram, we illustrate with $p = 2$. Choose a level 0 path $C$. Projecting onto these coordinates is a surjective homomorphism $\mathbf{W}_{\mathbb{Z}_p}(k) \to \mathbf{W}(k)$. The image is a domain, so the kernel is a prime ideal $p_C$. 
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There is a natural family of prime ideals in $\mathbb{W}_{\mathbb{Z}_p^2}(k)$. In the diagram, we illustrate with $p=2$. Choose a level 0 path $C$.

The intersection of the prime ideals described so far consists of the vectors whose coordinates are zero for any $\mathbb{Z}_p^2$-set in level 0. Since the only nilpotent in $\mathbb{W}_{\mathbb{Z}_p^2}(k)$ is 0, the intersection of all prime ideals in $\mathbb{W}_{\mathbb{Z}_p^2}(k)$ is $\{0\}$. There must be more prime ideals.
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More about primes

Set \( J_1 = \bigcap_C p_C \) the intersection of all the primes arising from projecting along rooted chains \( C \).
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More about primes

Set \( J_1 = \bigcap_C p_C \) the intersection of all the primes arising from projecting along rooted chains \( C \). So \( J_1 \neq 0 \), but taking the quotient \( \mathcal{W}_G(k)/J_1 \) restricts focus to the bottom portion of the diagram. In this quotient, are there more prime ideals?
Primes in $W_{\mathbb{Z}_p^2}(k)/J_1$

One may think of $W_{\mathbb{Z}_p^2}(k)/J_1$ as the Witt vectors associated to the tree:

![Tree Diagram]
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This ring $W_{\mathbb{Z}_p^2}(k)/J_1$ is still not coherent (and therefore not noetherian) and is still reduced. The proofs over $W_{\mathbb{Z}_p^2}(k)$ go through on the quotient.
Primes in $W_{\mathbb{Z}_p^2}(k)/J_1$

This viewpoint allows one to embed $W_{\mathbb{Z}_p^2}(k)/J_1$ into a more 'familiar' ring.
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![Diagram showing the boundary of the tree $T$]
This viewpoint allows one to embed $W_{Z_p^2}(k)/J_1$ into a more 'familiar' ring. The boundary of this tree, $T$, i.e., the space of all rooted paths is known to be $\mathbb{P}^1(\mathbb{Q}_p)$.

From a Witt vector $a \in W_{Z_p^2}(k)/J_1$, we can associate a function on $\partial T$ with values in $W(k)$. One can check that this gives an injective ring homomorphism $W_{Z_p^2}(k)/J_1$ into $\text{Fun}(\mathbb{P}^1(\mathbb{Q}_p), W(k))$. 
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So, contracting along this homomorphism gives new primes. Contracting the maximal ideals consisting of functions vanishing at a point recovers the primes \( p_C \) we found before.
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So, contracting along this homomorphism gives new primes. Contracting the maximal ideals consisting of functions vanishing at a point recovers the primes $p_C$ we found before.

And some we didn’t, like using common ultrafilter constructions.
Nonreduced Examples of $W_G(k)$

When $G$ is an infinite pro-$p$ group and $k$ is a field of characteristic $p$, $W_G(k)$ might be nonreduced.

**Theorem (M)**

- If $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}_p$, then $W_G(k)$ is nonreduced for all $k$ of characteristic $p$ when $p \neq 2$.
- When $G = D_{2\infty} = \lim_{\leftarrow} D_{2n} = \mathbb{Z}_2 \rtimes \{\pm 1\}$, $W_G(k)$ is nonreduced if $k$ has characteristic 2.
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The proofs of both cases use recursive computations to show an explicit nonzero Witt vector in $\mathcal{W}_G(\mathbb{F}_p) \subset \mathcal{W}_G(k)$ has square 0.
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The argument in the first case uses $p \neq 2$ in an important way, although the ring may be nonreduced when $p = 2$. 
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The ring $\mathbf{W}_{D_{2\infty}}(k)$ when $k$ has characteristic $2$ can also be shown to be not coherent.
Thank you.
Questions?


L.E. Miller, “A Witt-Burnside ring attached to a pro-dihedral group”, submitted.

